Closed-form Formula for Fibonacci Sequence

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Given Fibonacci sequence, 1,2,3,5,8,…, with the recursive formula $F_{n+1} = F_n + F_{n-1}$, where $F_1 = 1$ and $F_2 = 2$. Let $F_n = t^n$ and use the recursive formula to obtain

$$t^{n+1} = t^n + t^{n-1} \Longrightarrow t^2 = 1 + t \Longrightarrow t_1 = \frac{1+\sqrt{5}}{2}, t_2 = \frac{1-\sqrt{5}}{2}$$

Since the recursive formula F_n follows the linear form, we can get $F_n = C_1 * t_1^n + C_2 * t_2^n$ where C_1 and C_2 are two Constant parameters which are determined by the initial/boundary conditions. Bring in the condition $F_1 = 1$ and $F_2 = 2$, we can get $C_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$ and $C_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$. Finally, we know

$$F_n = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

This method is applicable to those linear form of recursive formula for sequence.

Basically, assume a given sequence a_n is defined by a recursive formula, which can be mapped to a quadratic equation $A * t^2 + B * t + C = 0$ using the above method. There are three cases:

Case 1: There are 2 different real roots t_1 and t_2, then

$$a_n = C_1 * t_1^n + C_2 * t_2^n$$

Case 2: There is only 1 real root t (collision), then

$$a_n = C_1 * t^n + C_2 * n * t^n$$

Case 3: There are 2 imaginary roots t_1 and t_2, then

$$a_n = C_1 * t_1^n + C_2 * t_2^n$$

Use the Euler formula $e^{ix} = \cos x + i \sin x$ and keep the real part, you can get the final solution. C_1 and C_2 are determined by the boundary conditions. For high order algebraic equation, the closed-form solution can be found only for degree =2,3,4. The same method can be applied.